“Spectral Asymptotics of the Neumann Laplacian on Open Sets with Fractal Boundaries”

Juan Pablo Pinasco

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SPECTRAL ASYMPTOTICS OF THE NEUMANN LAPLACIAN
ON OPEN SETS WITH FRACTAL BOUNDARIES

JUAN PABLO PINASCO

Abstract. We study the remainder estimate for the asymptotics of the number of eigenvalues of the Neumann laplacian in a bounded open set \( \Omega \in \mathbb{R}^n \) with fractal boundary. We improve the previous results, showing that the Minkowski dimension and content should be replaced by the interior Minkowski dimension and content.

1. Introduction

Let \( \Omega \) be an open bounded set in \( \mathbb{R}^n \) with finitely many connected components. We consider the following eigenvalue problem:

\[
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega \\
\frac{\partial u}{\partial \eta} = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( \frac{\partial u}{\partial \eta} \) denotes a "normal derivative" along \( \partial \Omega \). We interpret problem (1.1) in the variational sense, i.e., we say that \( \lambda \) is an eigenvalue if there exists a nonzero \( u \in H^1(\Omega) \) satisfying

\[
\int_{\Omega} \nabla u \nabla v = \lambda \int_{\Omega} uv \quad \forall v \in H^1(\Omega)
\]

We assume through the paper that the spectrum is discrete. We will discuss below some hypothesis on \( \Omega \) to ensure that the spectrum is a countable set without accumulation points. Let \( \{\lambda_j\}_j \) be the eigenvalues of the Neumann laplacian where \( 0 = \lambda_1 < \lambda_2 \leq \ldots \) (repeated according to their multiplicity). Let \( N(\lambda) = \#\{j : \lambda_j < \lambda\} \) be the associated spectral counting function. We are interested in the asymptotic behaviour of \( N(\lambda) \).

When \( \partial \Omega \) satisfies the so-called C-condition (see below), Metivier [Met1] showed that

\[
N(\lambda) = (1 + o(1))\varphi(\lambda)
\]

as \( \lambda \to \infty \). Here, \( \varphi(\lambda) = (2\pi)^{-n}\omega_n|\Omega|_n \lambda^{n/2} \), \( |A|_n \) denotes the \( n \)-dimensional Lebesgue measure (or volume) of \( A \subset \mathbb{R}^n \), and \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \). For smooth boundaries and under some other geometric assumptions ([Iv, Ph]), one has a second term when \( \lambda \to \infty \):

\[
N(\lambda) = \varphi(\lambda) + c_n|\partial \Omega|_{n-1} \lambda^{(n-1)/2} + o(\lambda^{(n-1)/2}),
\]

where \( c_n \) is a constant which depends only on \( n \).

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For the Dirichlet laplacian, in 1979 M. Berry [Br] made the following conjecture for $\partial \Omega$ with Hausdorff fractal dimension $h$:

$$N(\lambda) = \varphi(\lambda) - c_{h,n} \mu_h(\partial \Omega) \lambda^{h/2} + o(\lambda^{h/2}),$$

(1.5)

The conjecture cannot be true with $h$ being the Hausdorff dimension, [B-C], and the authors suggested to replace $h$ by $D_M$, the Minkowski (or box) dimension of the boundary.

In [Lap], the following asymptotic development was proved for the Dirichlet laplacian:

$$N(\lambda) = \varphi(\lambda) - c_{D_M,n} M_{D_M}(\partial \Omega) \lambda^{D_M/2} + o(\lambda^{D_M/2}),$$

(1.6)

as $\lambda \to \infty$, with $d$ the interior Minkowski dimension.

For the Neumann laplacian, Lapidus obtained a similar result replacing $d$ by $D_M$. This result was proved with a suitable extension of the Dirichlet-Neumann bracketing techniques, combined with precise estimates on the growth of the number of cubes in the tessellations of $\Omega$. Moreover, he conjectured:

**Conjecture 1.1.** Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ with boundary $\partial \Omega$ satisfying either the C-condition or the "extension property". Assume that $\partial \Omega$ is Minkowski measurable and that $D_M$ belongs to the open interval $(n-1,n)$. Then, for the Neumann laplacian, we have:

$$N(\lambda) = \varphi(\lambda) + c_{D_M,n} M_{D_M}(\partial \Omega) \lambda^{D_M/2} + o(\lambda^{D_M/2}),$$

(1.7)

where $c_{D_M,n}$ is a positive constant depending only on $n$ and $D_M$, and $M_{D_M}(\partial \Omega)$ denotes the Minkowski measure of $\partial \Omega$.

In this paper we show that $D_M$ should be replaced by $d$, the interior Minkowski dimension. Namely, we prove that

$$N(\lambda) = \varphi(\lambda) + O(\lambda^{d/2})$$

(1.8)

as $\lambda \to \infty$, as in the Dirichlet problem. We believe that the method used in [Lap] suggest the Minkowski dimension as the parameter involved in the spectral asymptotic. Instead of the Dirichlet-Neumann bracketing, we use the wave equations techniques. Our proof follows the ideas of Hormander [Ho] and Guillemin, [Gui].

The paper is organized as follows: In section 2, we introduce the necessary notation and definitions, and we give a precise formulation of our results. Finally, in section 3 we prove the main theorem.

### 2. Hypothesis and main results.

Let $A_\varepsilon$ denote the tubular neighborhood of radius $\varepsilon$ of a set $A \subset \mathbb{R}^n$, i.e.,

$$A_\varepsilon = \{ x \in \mathbb{R} : \text{dist}(x,A) \leq \varepsilon \}$$

(2.1)

We define the Minkowski dimension $D_M$ of $\partial \Omega$ as

$$D_M = Dim_M(\partial \Omega) = \inf \{ \delta \geq 0 : \lim_{\varepsilon \to 0^+} \sup_{\Omega_\varepsilon} \varepsilon^{-(n-\delta)} |(\partial \Omega)_\varepsilon|_n = 0 \}$$

(2.2)

In a similar way, we define the interior Minkowski dimension as

$$d = dim(\partial \Omega) = \inf \{ \delta \geq 0 : \lim_{\varepsilon \to 0^+} \sup_{\Omega_\varepsilon} \varepsilon^{-(n-\delta)} |(\partial \Omega)_\varepsilon \cap \Omega_\varepsilon|_n = 0 \}$$

(2.3)
It is evident that $d \leq D_M$. Moreover, even when $|\partial \Omega|_n$ is positive, the interior Minkowski dimension can be strictly lower than $n$.

We define the interior Minkowski content of $\partial \Omega$ as the limit (whenever it exist):

$$M_{\text{int}}(\partial \Omega, d) = \lim_{\varepsilon \to 0^+} \varepsilon^{-(n-d)} |(\partial \Omega) \cap \Omega|.$$

Respectively, $M_{\text{int}}^*(\partial \Omega, d)$ denotes the $d-$dimensional upper (lower) interior Minkowski content, replacing the limit in (2.4) by an upper (resp., lower) limit.

It is not known a geometrical condition on $\Omega$ equivalent to the discreteness of the spectrum, however, we can impose the following condition in $\Omega$ (see [Met1]) in order to ensure it:

**Definition 2.1.** The open set $\Omega$ satisfies the C-condition if there exist positive constants $\varepsilon_0$, $M$, $t_0$ with $\varepsilon_0 M < t_0$, an open cover $\{\Omega_j\}_{1 \leq j \leq N}$ of $\partial \Omega$, and nonzero vectors $h_j$ ($1 \leq j \leq N$) in $\mathbb{R}^n$ such that

$$\forall j, \forall (x, y) \in \Omega_j \times \Omega_j \text{ with } |x - y| < \varepsilon_0,$$

$$\forall t \in \mathbb{R} \text{ with } M|x - y| \leq t \leq t_0, \text{ the line segments } [x, x + th_j], [y, y + th_j] \text{ and } [x + th_j, y + th_j] \text{ are all contained in } \Omega.$$

Condition C holds in many cases, for instance if $\Omega$ satisfies the so-called "segment condition" or the "cone condition", or if $\partial \Omega$ is locally Lipschitz. See [F-M] or [Met2] for details.

Another condition is the "extension property" (i.e., the existence of a continuous linear extension map from $H^m(\Omega)$ to $H^m(\mathbb{R}^n)$). Under such hypothesis, it is possible to ensure that the embedding mapping from $H^1(\Omega)$ to $L^2(\Omega)$ is compact and as a consequence that the spectrum is discrete. Sufficient condition for the domain $\Omega$ to satisfy the "extension property" are obtained by Jones in [Jn].

Let us state our main results:

**Theorem 2.2.** Let $\Omega \in \mathbb{R}^n$ be an open, bounded set, and $d \in [n-1, n]$ such that $M_{\text{int}}^*(\partial \Omega, d) < +\infty$. We have the following estimates

i) If $d \in (n-1, n]$, then:

$$N(\lambda, \Omega) = \varphi(\lambda) + O(\lambda^{d/2})$$

when $\lambda \to +\infty$.

ii) If $d = n-1$, then

$$N(\lambda, \Omega) = \varphi(\lambda) + O(\lambda^{d/2} \ln(\lambda))$$

when $\lambda \to +\infty$.

The next result deals with the degenerate case when $M_{\text{int}}^*(\partial \Omega, d) = +\infty$:

**Corollary 2.3.** Let $\Omega \in \mathbb{R}^n$ be an open, bounded set. Let $d \in [n-1, n]$ be the interior Minkowski dimension of $\partial \Omega$. Then we have the following remainder estimates

$$N(\lambda, \Omega) = \varphi(\lambda) + o(\lambda^{D/2})$$

when $\lambda \to +\infty$, for all $D > d$.

**Remark 2.4.** The same asymptotics are valid changing the Neumann boundary condition by a mixed Dirichlet-Neumann boundary condition, i.e., imposing $u = 0$ in $\Gamma \subset \partial \Omega$ and $\frac{\partial u}{\partial n} = 0$ in $\partial \Omega \setminus \Gamma$. 
Remark 2.5. When \( d = n \) and \( M_{nt}(\partial \Omega, d) = +\infty \), the Weyl’s term \( \varphi(\lambda) \) may be changed. See [Pi] for related examples.

3. Proof of the main theorem

We will need in the proof the following tauberian theorem (we refer the reader to [Gui] in [Ho2] for a proof):

**Theorem 3.1.** (Hormander) Let \( \gamma_n : [0, \infty) \to \mathbb{R} \) be the function \( \gamma_n(\lambda) = \lambda^n \) and let \( m : [0, \infty) \to \mathbb{R} \) be any non-decreasing function of polynomial growth with \( m(0) = 0 \). Suppose the cosine transforms of \( \frac{dn}{d\lambda}, \frac{dm}{d\lambda} \) are equal on the interval \( |t| \leq \delta \).

Then

\[
|m(\lambda) - \lambda^n| \leq C_n \left( \frac{1}{\delta^n} + \frac{\lambda^{n-1}}{\delta} \right)
\]

for \( \lambda > 0 \), where \( C_n \) is a universal constant depending only on \( n \).

Let \( \varphi_1, \varphi_2, \ldots \) be the normalized eigenfunctions corresponding to the eigenvalues \( \lambda_1, \lambda_2, \ldots \). Let \( E(\lambda) \) be the orthogonal projection on the space spanned by the eigenfunctions corresponding to \( \lambda_i \leq \lambda^2 \). The Schwartz kernel of \( E(\lambda) \) is the spectral function

\[
e(x, y; \lambda) = \sum_{\lambda_i \leq \lambda^2} \varphi_i(x) \varphi_i(y).
\]

Obviously, \( N(\lambda) = \int e(x, x; \lambda) dx \), and it is easy to see, using the maximum principle for parabolic equations, that \( e(x, x; \lambda) \leq C \lambda^n \forall x \in \Omega \) (see [Gui] for a proof).

Now we prove Theorem 2.2.

**Proof.** Following Hormander, the cosine transform

\[
u(x, y; t) = \int_0^\infty \cos(\lambda t) \frac{d}{d\lambda} e(x, y; \lambda) d\lambda
\]

is the fundamental solution of the wave equation (see [R-T] for arbitrary open sets):

\[
\begin{cases}
\frac{\partial^2}{\partial t^2} u(x, y; t) = \Delta u(x, y; t) & \text{in } \Omega \\
\frac{\partial u(x, y; t)}{\partial n} = 0 & \text{on } \partial \Omega
\end{cases}
\]

The free space solution \( u_0(x, y; t) \) of 3.4 has a closed form expression,

\[
u_0(x, y; t) = \frac{\omega_n}{(2\pi)^n} \int_0^\infty \cos(st) \frac{d}{ds} s^n ds
\]

where \( \omega_n = \text{vol}(S^{n-1})/n \), and they must agree with the solution \( u(x, y; t) \) if \( t < \text{dist}(y, \partial \Omega) \), as a consequence of the wave’s finite speed of propagation.

Using the Tauberian Theorem (3.1), we obtain:

\[
|e(x, x; \lambda) - \frac{\omega_n}{(2\pi)^n} \lambda^n| \leq C_n \left( \frac{1}{\delta^n} + \frac{\lambda^{n-1}}{\delta} \right)
\]
for $\text{dist}(x, \partial \Omega) > \delta$. We can estimate $|N(\lambda) - \frac{\omega_n}{(2\pi)^n} |\Omega| \lambda^n|$ by

$$\int_\Omega |e(x, x; \lambda) - \frac{\omega_n \lambda^n}{(2\pi)^n} dx| \leq \int_{\Omega \cup U} |e(x, x; \lambda) - \frac{\omega_n \lambda^n}{(2\pi)^n} dx + \int_U C_n \left( \frac{1}{\delta^n} + \frac{\lambda^{n-1}}{\delta} \right) dx$$

where $U = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > 1/\lambda \}$.

The second integral is majorized by

$$(3.7) \quad C \int_{1/\lambda}^t \left( \frac{1}{\delta^n} + \frac{\lambda^{n-1}}{\delta} \right) d\delta$$

for $t$ large enough, which is of order $O(\lambda^{n-1} \ln(\lambda))$.

To estimate the other integral, we note that, when $\partial \Omega$ is sufficiently smooth, it is of order $O(\lambda^{n-1})$ because we have $e(x, x; \lambda) \leq C \lambda^n$ uniformly on $\Omega$, and $|\Omega \setminus U|_n$ is approximately $1/\lambda$ times the perimeter.

In our situation, we can use the interior Minkowski content of $\partial \Omega$ to obtain:

$$(3.8) \quad |\Omega \setminus U|_n = |\partial \Omega|_1 \cap \Omega|_n \leq A \left( \frac{1}{\lambda^{n-\sigma}} \right)$$

with $A = \sup_{(\varepsilon<1)} \varepsilon^{-(n-d)} |\partial \Omega_\varepsilon \cap \Omega| < \infty$. It follows immediately that the integral is of order $O(\lambda^d)$, which is greater than the other integral if and only if $d > n - 1$. □

Remark 3.2. Note that Corollary (2.3) follows from Theorem (2.2), since according to the definition of Minkowski dimension, $D > d$ implies $M^1_{\text{int}}(\partial \Omega, d) < +\infty$.

Taking $d'$ with $d < d' < D$, we can replace the estimate $O(\lambda^{d'/2})$ obtained from Theorem (2.2) by $O(\lambda^{d'/2})$. Further, when $d = n - 1$ we can substitute $O(\lambda^{d'/2} \ln(\lambda))$ by $o(\lambda^{d'/2})$.

References


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JUAN PABLO PINASCO
UNIVERSIDAD DE SAN ANDRES
VITO DUMAS 284 (1684), PROV. BUENOS AIRES, ARGENTINA.
E-MAIL: JPINASCO@DM.UBA.AR